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On Resonance in Infinite Gratings of Cylinders

S. N. KARP and J. RADLOW

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Abstract

The diffraction by a grating is examined (for spacing large compared to wavelength and dimension of grating element) for wavelengths in the neighborhood of the "Rayleigh" wavelengths. The shape of the elements, and their size in wavelengths is unrestricted. The results, including the effect of interaction, are expressed in terms of quantities relating to single scattering. Some properties of certain determinants formed from single scattered amplitudes are derived. The results are compared with those obtained by other authors, using various restrictions on the parameters.

Table of Contents

		Page
1.	Introduction	1
2.	Preliminary results	3
3.	The case of resonance	6
4.	Remarks on energy conservation	13
5.	Comparison with other theories	14
	Appendices	22
	References	40

1. Introduction

The purpose of the present work is to present the results of an investigation of the behavior of diffraction gratings of cylinders for wavelengths neighboring certain critical wavelengths, the so-called " Rayleigh wavelengths ". The analysis is based on the results of a previous investigation [1], [2], in which expressions were derived for the Fourier amplitudes of the diffracted field of an grating of arbitrary cylinders, subject to the restrictions that the spacing is large as compared to cylinder dimension and wavelength, and that the Rayleigh wavelengths are excluded. The expressions there obtained for the Fourier amplitudes only involved quantitites associated with a single scatterer, in spite of the fact that interactions were taken into account. The principal result of the present paper is the extension of the conclusions of [2] to the case of critical wavelengths, in which the significance of interaction is maximal.

The required results of [1] and [2] are recapitulated in Section 2 of the present work. They form a special case of a general theory [3] for the calculation of the field scattered by a combination of obstacles in terms of the scattering patterns of the obstacles in isolation. Experience with the case of diffraction by a pair of half-planes (i.e., a slit) [4] indicates that the accuracy of this kind of calculation is good down to spacing of one wavelength.

In Section 3 of the present paper we discuss the behavior of the Fourier amplitude of the grating field in the limit in which the critical or Rayleigh wavelengths are approached. In this limit the " single scattering" result diverges, while the result given in Section 2 is indeterminate. However, the limit is evaluated explicitly in the present work. We also include in Section 3 a discussion of certain determinants involving the scattering amplitudes of a single cylinder; the behavior of these determinants is crucial for the limiting process in question.

Section 4 is concerned with a brief discussion of energy conservation. It is shown that the " single scattering" result conserves energy with an error which, while small for general large spacings, contains a factor which diverges for critical spacings. On the other hand, our results are free of this defect.

In Section 5 we compare our results with those of previous investigators. Most previous work has been based on other types of restrictions than our own. However, Twersky[5], [6], who treated the case of circular cylinders for non-critical spacings, obtained equivalent results for the contributions of the individual cylinders in that case, for certain angles of incidence; and these results were extended by Magnus (in Appendix to [5]) to arbitrary angles of incidence subsequent to the developments in [1].

A more interesting comparison is made possible by considering the application of our results to the reflection grating, which is the seat of the celebrated " Wood's Anomalies" discussed by Wood[7], Rayleigh[8], Fano[9] and Strong[10]. A definitive theory has been given by Artmann[11]. Since the only restriction of Artmann's analysis is that the corrugations be small compared to wavelength, a comparison with our theory is possible in the case in which corrugations are small and the spacing is large. We specialize our results to small corrugations in Section 5, and, after giving explicit approximate formulas for the scattering amplitudes of a thin strip in the case when the magnetic vector is parallel to the generators of the grating, we are able to show agreement with the results of Artmann.

On the other hand it is pointed out that both for the resonance and non-resonance cases our results are in disagreement with results given in a new report by Twersky[12]. The nature of the discrepancy is explained and possibilities for its removal are suggested.

Certain calculations, omitted in the main body of the text in order to keep the argument concise, are carried out in detail in the appendices.

2. Preliminary results

It is well known that in the far field every outgoing wave function takes the form[*]

$$(1) \qquad H_0(kr)\, f(\theta) \cong \sqrt{\frac{2}{\pi kr}}\, \exp(ikr - i\,\frac{\pi}{4}) \cdot f(\theta) \quad ,$$

where r and θ are polar coordinates. Also, in a problem dealing with diffraction by a combination of obstacles, Green's theorem shows that we can regard the scattered field as composed of a sum of outgoing wave functions, each regular in the exterior of one of the scattering bodies. Suppose the bodies in question are sufficiently far apart for all dimensions of each body, as well as the wavelength, to be small compared to the distance to any other body in the array. Then it can be shown from (1) that in the neightborhood of any body in the configuration the total field is approximately equal to the incident field plus the field scattered by the body in question plus a sum of plane waves arriving from the directions of the other bodies. [The complex amplitudes of the latter plane waves are at first unknown.] Consequently the actual field contributed by each body is approximately an unknown linear combination of the fields which that body would scatter when responding to the plane waves ih question in isolation. The necessary coefficients can then be calculated by a consistency argument. For details in the case of $E||$, see [2], [3] (the argument is easily seen to apply for arbitrary boundary conditions, however).

The theory in question can be applied to an infinite grating of cylinders. Before stating the results we recall some formulas from the general theory of infinite gratings.

We consider a grating of congruent cylinders lying in the plane $y = 0$. If a plane wave is incident at the angle ϕ, so that

---

[*] This is the two-dimensional result. Time dependence $\exp(-i\omega t)$ is assumed.

$$(2) \qquad u_{inc} = \exp\left[ik(x \cos \phi + y \sin \phi)\right],$$

and if the distance between corresponding points of the grating is d, then the fields contributed by the various cylinders differ among each other only by a translation and by a phase factor $\exp[iknd \cos \phi]$, so that the total field may be written

$$(3) \qquad u = u_{inc} + \sum_{-\infty}^{\infty} \exp\left[iknd \cos \phi\right] F(x - nd, y) .$$

Suppose further that the far field form of $F(x,y)$ is given by

$$(4) \qquad F(x,y) \sim \sqrt{\frac{2}{\pi kr}} \exp(ikr - i \frac{\pi}{4}) f(\theta) ,$$

where $x = r \cos \theta$, $y = r \sin \theta$. Then in consequence of the regularity of the field in the exterior of a horizontal strip containing the grating elements, the total field can be written as follows (see [2]):

$$(5) \qquad u = u_{inc} + \sum_{-\infty}^{\infty} \frac{2}{kd \sin \theta_\nu} \cdot f(\pm \theta_\nu) \exp\left[ikr \cos(\theta \mp \theta_\nu)\right]$$

except for values of y in horizontal strips containing the grating elements. The upper sign is to be taken when y is above the strip. In this exact formula the spectral angles θ_ν are given by the formula

$$k \cos \theta_\nu = k \cos \phi + 2\pi\nu/d ,$$

$$k \sin \theta_\nu = \sqrt{k^2 - (k \cos \phi + 2\pi\nu/d)^2} .$$

The infinite grating problem is thus equivalent to the problem of determining the values of the function $f(\theta)$ for $\theta = \theta_\nu$. A finite number of the quantities θ_ν are real; these lead to propagating waves in (5) (i.e., orders of the spectrum). The function $f(\theta)$ can be expressed in terms of the boundary values of $F(x,y)$ by quadratures which are meaningful also for complex θ. We can now state our preliminary formulas, with which we operate in the balance of this paper. We have found, using the type of analysis described in the foregoing paragraph, that

$$f(\theta) = f_I(\theta) + A_L f(0) f_L(\theta) + A_R f(\pi) f_R(\theta) , \tag{7}$$

$$F(x,y) = F_I(x,y) + A_L f(0) F_L(x,y) + A_R f(\pi) F_R(x,y) , \tag{7a}$$

where the functions $f_I(\theta)$, $f_L(\theta)$ and $f_R(\theta)$ are those describing the far field of a single cylinder in the grating, when excited respectively by the incident field, a plane wave coming from $x = -\infty$, and a plane wave coming from $x = +\infty$. The functions $F_I(x,y)$, etc., are the corresponding scattered fields defined for all x and y. The constants A_L and A_R are given by the formulas

$$\begin{aligned} A_L &= \sqrt{\frac{2}{\pi kd}} \exp\left[-i \frac{\pi}{4}\right] \sum_1^\infty \exp\left[inkd\,(1 - \cos \phi)\right] / \sqrt{n} , \\ A_R &= \sqrt{\frac{2}{\pi kd}} \exp\left[-i \frac{\pi}{4}\right] \sum_1^\infty \exp\left[inkd\,(1 + \cos \phi)\right] / \sqrt{n} . \end{aligned} \tag{8}$$

The quantities $f(0)$, and $f(\pi)$ (which are obtained from (7) by setting $\theta = 0$ and $\theta = \pi$ respectively, and solving the resulting pair of simultaneous equations), are given by the following formulas

$$\begin{aligned} f(0) &= \left[f_I(0) - A_R \alpha_1\right] \left\{1 - \left[A_L f_L(0) + A_R f_R(\pi)\right] + A_R A_L \alpha_3\right\}^{-1} \\ f(\pi) &= \left[f_I(\pi) - A_L \alpha_2\right] \left\{1 - \left[A_L f_L(0) + A_R f_R(\pi)\right] + A_R A_L \alpha_3\right\}^{-1} , \end{aligned} \tag{9}$$

in which α_1, α_2, α_3 are given by the formulas

(10) $$a_1 = \begin{vmatrix} f_I(0) & f_R(0) \\ f_I(\pi) & f_R(\pi) \end{vmatrix}, \quad a_2 = \begin{vmatrix} f_L(0) & f_I(0) \\ f_L(\pi) & f_I(\pi) \end{vmatrix}, \quad a_3 = \begin{vmatrix} f_L(0) & f_L(\pi) \\ f_R(0) & f_R(\pi) \end{vmatrix}.$$

This result, which is obtained in [2], holds provided that k, d, and ϕ are such that there is no positive or negative integer ν for which $\sin \Theta_\nu$ vanishes. This implies that both A_L and A_R converge. Then A_L and A_R are small quantities for sufficiently large kd, so that the leading term of (7) is $f_I(\theta)$. The approximation $f(\theta) \sim f_I(\theta)$ thus expresses the natural assumption that each grating element responds as if in isolation, when the spacing is very large.

3. The case of resonance

Our task in the present work is to discuss the behavior of the solution in the neighborhood of those frequencies or angles of incidence that make $\sin \Theta_\nu = 0$. We shall refer to such a situation as a " resonance", and shall discuss the limiting value of the result (7), as well as the behavior in the neighborhood of the limit. [We shall not discuss the mathematical question of the existence and properties of the solution at precisely critical frequencies. Such a solution may well differ from the limit of the non-critical solution which, it seems to us, is the observable quantity here.]

First we shall discuss the limiting value of the solution given by (5) and (7).

To do so we note that if ν is such that $\sin \Theta_\nu$ tends to zero, then the denominator of the ν-th term of (5) goes to zero. The corresponding exponential factor in this term becomes $\exp(\pm ikx)$. This analytical phenomenon is well known since the time of Rayleigh, who pointed out the connection between grating resonances and the case when a spectral order passes off tangentially. Of course, the approximation $f(\theta) \sim f_I(\theta)$ breaks down completely in this case. However, we shall show that the use of (7) leads to an intelligible result. Our task will be to compute the limit of $f(\theta_\nu)/\sin\theta_\nu$ as $\theta_\nu \to 0, \pi$. In addition it will be noted that all other terms of the series (5) must be re-examined in this case, since one or both of the quantities A_L and A_R diverge, and the quantities $A_L f(0)$ and $A_R f(\pi)$ become indeterminate.

In order to proceed it is necessary to scrutinize the manner of divergence of A_L and A_R. This divergence can occur in several ways (cf. (8)). (a): $kd(1-\cos\phi)$ may approach an integral multiple of 2π, or (b): $kd(1+\cos\phi)$ may approach such a value, or (c): both case (a) and case (b) may occur simultaneously. (For the last mentioned case to occur $\tan^2(\phi/2)$ must necessarily be rational.) Suppose therefore that there exist positive integers p_a, p_b such that either

$$(11a) \qquad kd(1-\cos\phi) = 2p_a\pi + \delta_a \qquad \delta_a \to 0 ,$$

or

$$(11b) \qquad kd(1+\cos\phi) = 2p_b\pi + \delta_b , \qquad \delta_b \to 0$$

or both. In case (a) the denominator of the term $\nu = p_a$ vanishes in (5) and also A_L diverges. In case (b) it is the term $\nu = -p_b$ which is critical while A_R diverges. It can be shown* by comparison with the binomial expansion of $(1-z)^{-1/2}$ that for small $\delta_{a,b}$ we have

$$(12) \qquad A_{L,R} \to 2/(2kd\delta_{a,b})^{1/2} ,$$

where the root is positive or positive imaginary. In case (c), (11) and (12) show that

$$(13) \qquad \lim A_L/A_R = \lim \sqrt{\delta_b/\delta_a} \quad .$$

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* See Appendix I-A.

If we keep ϕ fixed and let k approach a critical value, then the limit is cot $\phi/2$.* If k is a suitably fixed value and ϕ approaches a correspondingly critical value, the limit is + i. On the other hand, (6) allows us to express k sin Θ_ν approximately in terms of δ_a and δ_b, when $\nu = p_a$ and $\nu = -p_b$ respectively. Introducing $k_\nu = k \sin \phi_\nu$, we find

$$(14) \qquad k_{p_a} \rightarrow (2kd\delta_a)^{1/2} / d; \qquad k_{-p_b} \rightarrow (2kd\delta_b)^{1/2} / d$$

so that

$$(15) \qquad \begin{aligned} &\lim k_{p_a} A_L = \lim k_{-p_b} A_R = \frac{2}{d} , \\ &\lim k_{p_a} A_R = \frac{2}{d} \lim (A_R/A_L) , \qquad \lim k_{-p_b} A_L = \frac{2}{d} \lim (A_L/A_R) . \end{aligned}$$

Since A_L alone diverges in case (a), we can calculate the limit $A_L f(0)$ directly from (9), and then (7) becomes

$$(16) \qquad f(\theta) = f_I(\theta) + \frac{f_I(0) - A_R\alpha_1}{A_R\alpha_3 - f_L(0)} f_L(\theta) - \frac{A_R\alpha_2}{A_R\alpha_3 - f_L(0)} f_R(\theta) .$$

We noticed earlier that for large and non-critical kd the single scattering hypothesis $f(\theta) = f_I(\theta)$ supplies a first approximation to (7). This is no longer the case as a critical value is approached. Indeed, neglecting the correction supplied by terms of order $A_R = (1/\sqrt{kd})$ in (16) we find

$$(17) \qquad f(\theta) \rightarrow f_I(\theta) - \frac{f_I(0)}{f_L(0)} f_L(\theta) ,$$

which differs from $f_I(\theta)$ by a finite amount. In case (b) a similar result is obtained. In (17) it should be noted that the denominator is the forward scattering amplitude of an isolated cylinder. Its negative real part gives the scattering cross section, by a well-known theorem. The vanishing of the cross section implies that

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* See Appendix I-B for the evaluation of $\lim(A_L/A_R)$ in the various cases; also for the evaluation of the limits in equations (14) and (15).

$$\int_0^{2\pi} |f_L(\theta)|^2 \, d\theta \equiv 0 .$$

This in turn implies $f_L(\theta) \equiv 0$ for all θ, whence also the vanishing (everywhere) of the field scattered by the cylinder when excited by the incident field exp(ikx). The incident plus scattered field could not then combine so as to fulfill homogeneous boundary conditions. Consequently $f_L(0)$ cannot vanish.

We now consider case (c). In this case we have to consider two possibilities: either α_3 vanishes, or it does not. Both possibilities could actually occur. It is easily verified, by calculating the cross section explicitly, that for the case of a circular cylinder α_3 cannot vanish. On the other hand, for a strip of zero thickness along the y-axis the symmetry of the scattered field shows that α_3 does vanish.

It is of interest to discuss the determinants α_1, α_2 and α_3. These are properties of a single cylinder. It can be shown that when α_3 vanishes so do α_1 and α_2.* To see this define $v = f_L(0)\, F_R(x,y) - f_R(0)\, F_L(x,y)$, which is an outgoing wave function. Then form the quantity $E(v) \equiv \mathrm{Im} \oint v(\partial\bar{v}/\partial n)ds$ over the body. Then, using the boundary condition** for F_L and F_R, and the well-known expression for the far field amplitude as a surface integral, we find $E[v] = 4\mathrm{Re}\left\{\bar{f}_L(0)\alpha_3\right\}$. Consequently the vanishing of α_3 implies $E[v] \equiv 0$. Since v is an outgoing wave function, it now follows that v is identically zero. (The argument is that used in the proof of the uniqueness theorem for diffracted fields, which actually requires only the vanishing of the energy flux E(v) over the surface.)

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* See Appendix II.

** Thus, for $E||$, $F_R = - e^{-ikx}$ on the body.

Having now shown that $\alpha_3 \equiv 0$ implies $v \equiv 0$, consideration of the far field form of v shows that

$$f_L(0)f_R(\theta) - f_R(0)f_L(\theta) \equiv 0$$

for all θ. Now by the reciprocity theorem for plane waves, we easily see that $f_I(0) = f_R(\phi+\pi)$, $f_I(\pi) = f_L(\phi+\pi)$, and $f_L(0) = f_R(\pi)$. (Recall for example that $f_I(0)$ is the far field amplitude in the direction 0 produced by a plane wave incident in direction ϕ while $f_R(\phi+\pi)$ is the far field amplitude resulting in the direction $\phi+\pi$, when a plane wave is incident from the direction $\theta = 0$.) After these substitutions it follows directly that $\alpha_1 = 0$. The vanishing of α_1 implies (by direct substitution of $f_R(\pi)f_I(0)/f_R(0)$ for $f_I(\pi)$) that

$$\alpha_2 = \frac{f_I(0)\alpha_3}{f_R(0)} .$$

Consequently $\alpha_3 \equiv 0$ also implies $\alpha_2 = 0$, as was to be shown. [It seems worthwhile, in passing, to inquire as to when α_3 can vanish. The necessary and sufficient condition (for perfectly conducting bodies) in the case of E|| is that the boundary is composed of one or more vertical zero-thickness strips. In the H|| polarization any number of horizontal strips may be added. The spacing of the vertical strips must be according to half-integral multiples of the wavelength. Thus in the H|| case we can have a rectangle of suitable width. The back and forward scattered amplitudes agree for E||, and differ only in sign for H||. The result generalizes well-known symmetry properties of a screen of zero thickness. If a general homogeneous boundary condition is to be satisfied, one can derive a differential equation relating the equation of the curve and the coefficients in the boundary condition. These results all follow easily by applying the appropriate boundary condition. For the E|| case, for example, we have

$$v \equiv f_L(0)F_R - f_R(0)F_L = -\left[f_L(0)\, e^{-ikx} - f_R(0)\, e^{ikx}\right]$$

on the boundary. If $\alpha_3 = 0$, then v vanishes identically and hence x must be constant on the boundary. Taking the origin of x at a boundary point, we see $f_L(0) = f_R(0)$. Hence at other boundary points sin kx = 0, whence the spacing can be deduced. The sufficiency is shown by defining

$$W \equiv F_L - F_R \quad ,$$

say, in the case E||. Then W will be seen to vanish on the boundary and hence $W \equiv 0$, so that $F_L = F_R$. Consequently $f_L(\theta) = f_R(\theta)$ for all θ, whence it is easily shown that α_1, α_2 and α_3 vanish.]

Returning now to case (c) we first note the result when $\alpha_3 \neq 0$. Taking the limit in (7), with $f(0)$ and $f(\pi)$ given by (9), as both A_L and A_R approach ∞, we obtain

$$(18) \qquad f(\theta) = f_I(\theta) - \left[\alpha_1 f_L(\theta) + \alpha_2 f_R(\theta)\right] / \alpha_3 \quad .$$

On the other hand, if α_3 vanishes, then so do α_1 and α_2; and, using the relation between $f_L(\theta)$ and $f_R(\theta)$ we obtain in general

$$(19) \qquad f(\theta) = f_I(\theta) + \frac{\left[A_L + A_R\right]\left[f_I(0) f_L(\theta)\right]}{1 - \left[A_L + A_R\right] f_L(0)}$$

and therefore in the limit

$$(20) \qquad f(\theta) = f_I(\theta) - \frac{f_I(0)}{f_L(0)}\, f_L(\theta) \quad .$$

We have now calculated the limit which $f(\theta)$ approaches in cases(a), (b) and (c). This allows immediate calculation of the complex amplitudes of the non-critical spectra, i.e., the terms in (5) for which the denominator does not tend to zero. We still have to examine the terms $\nu = p_a$, and/or $\nu = -p_b$, where p_a and p_b are given by (11).

In case (a) we find*, using (15),

$$(21) \quad \lim \frac{2f(\theta_{p_a})}{kd \sin \theta_{p_a}} = \left\{ f_I(0) - A_R\alpha_1 - \frac{2}{kd}\left(f_I'(0) f_L(0) - f_I(0) f_L'(0) - A_R\beta'(0) \right) \right\} \left[A_R\alpha_3 - f_L(0) \right]^{-1}$$

where $-\beta(\theta) = \alpha_3 f_I(\theta) - \alpha_1 f_L(\theta) - \alpha_2 f_R(\theta)$ (note that $\beta(0) = 0$). Since kd is always large, this simplifies to

$$(22) \qquad \lim \frac{2}{kd} \frac{f(\theta_{p_a})}{\sin \theta_{p_a}} \rightarrow -\frac{f_I(0)}{f_L(0)} \quad .$$

The result (21) furnishes a correction to this formula.

Thus the total coefficient in the last spectrum tends to a finite quantity as this spectrum tends to the horizontal.

In case (b) we find, for the critical term $\nu = -p_b$, the corresponding approximate value $-f_I(\pi)/f_R(\pi) = -f_I(\pi)/f_L(0)$.

Turning now to case (c), consider first the general situation, i.e., $\alpha_3 \neq 0$. We find the coefficient of the term $\nu = p_a$ tends to

$$(23) \qquad \lim \frac{2f(\theta_{p_a})}{kd \sin \theta_{p_a}} = -\frac{\alpha_1}{\alpha_3} - \frac{2}{kd}\frac{\beta'(0)}{\alpha_3} \quad .$$

Similarly, for $\nu = -p_b$

$$\lim \frac{2}{kd} \frac{f(\theta_\nu)}{\sin \theta_\nu} = -\frac{\alpha_2}{\alpha_3} - \frac{2}{kd}\frac{\beta'(\pi)}{\alpha_3} ,$$

which is again a finite result.

As resonance is approached, in the case $\alpha_3 = \alpha_2 = \alpha_1 = 0$ we get instead, for $\nu = p_a$

$$\lim \frac{2}{kd} \frac{f(\theta_\nu)}{\sin \theta_\nu} = -\frac{f_I(0)}{f_L(0)\,|1+\sigma|}$$

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* See Appendix III.

plus a correction term in $(kd)^{-1}$. Here σ is the limit approached by the ratio A_R/A_L (cf. (13) and the subsequent text).

4. Remarks on energy conservation*

For any wave function that is regular in the exterior of the grating elements, the total energy passing through the boundaries of a typical cell will be equal to the energy passing through the surface of the typical cylinder contained in that cell, as can be seen from direct applications of Green's theorem. (Furthermore, due to the periodicity, the energy passing through the vertical boundaries of the cell is zero.)

Consequently, of course, energy is conserved in the correct solution just as well as in the incident field. For an incorrect solution, however, the degree of conservation of energy will be measured by the value of the surface integral of Poynting's vector, provided we are dealing with a suitably periodic wave function, which is regular in the exterior of the cylinders of the grating. Now, with regard to the single scattering solution, according to which each cylinder contributes to the field as if in isolation, it is easy to show that the boundary values are adopted with an error of order A_L, say. Consequently energy is conserved to order zero, except in the resonance case. In that case A_L diverges, so that energy is not conserved at all. [To see this, we simply note, for example, that a cylinder at a distance nd to the right of a given " central" cylinder contributes a field of the order of

$$H_o(nkd)\ f_I(\pi)$$

to the boundary values on the cylinder selected as "central".] On the other hand, in our analysis, the contribution of a single cylinder is given by formula (7a). It is easy to show that then the surface energy integral vanishes to order

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* See Appendix IV.

$(kd)^{-1/2}$ and that the error is not only of higher order in $(kd)^{-1}$ for a non-critical case, but it is also uniformly bounded, i.e., does not diverge as resonance is approached. Consequently in our result energy is conserved to the proper order, and this independently of the " resonance".

It should perhaps be added, however, that conservation of energy in itself does not seem to guarantee that the boundary conditions are even approximately fulfilled. From the vanishing of $\mathrm{Im} \int u\, \partial\bar{u}/\partial n\, ds$ over a surface, only the orthogonality of the function and its normal derivative follows. For example, the exact solution for a grating of perfect dielectric cylinders is certainly useless as a substitute for the solution of the corresponding problem of conducting cylinders. Yet it certainly fulfills conservation of energy. Consequently, while the energy law is useful as a check, proposed approximate solutions always require additional support.

5. Comparison with other theories

Since, as is well known, the theory of a grating of conducting cylinders implies a corresponding theory for the conducting reflection grating, it seems useful to compare the results so obtainable from our solution with results obtained by Artmann. Artmann's results are obtained under the assumption of protuberances small compared to wavelength, without restriction as to spacing. By specializing his results to the case of large spacing, and specializing our results to the case of thin cylinders, a valid comparison is therefore possible. However, we require explicit expressions for the quantities $f_L(0)$, etc., since Artmann's results are expressed in terms of the geometry of the surface. To this end, we have calculated, by the method of small perturbations, the pattern produced by diffraction by a thin strip of thickness $y = 2\varepsilon f(x)$ where $k\varepsilon$ is small*.

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* See Appendix V for this calculation.

We consider a symmetrical strip $y = \pm \varepsilon f(x)$, for $|x| < \ell$. The incident magnetic field has a single component, which is perpendicular to the xy-plane and is given by $F_{inc} = \exp(ikx)$. Then if we write

$$H(x,y) = \exp(ikx) + F_L(x,y) ,$$
$$F_L(x,y) = \varepsilon F_L^{(1)}(x,y) + \varepsilon^2 F_L^{(2)}(x,y) + \dots \tag{23}$$

we find

$$F_L^{(1)}(x,y) = \frac{k}{2} \int_{-\ell}^{\ell} e^{ikx_o} f'(x_o) H_o^{(1)} \left(k \sqrt{(x-x_o)^2 + y^2}\right) dx_o \ , \tag{24}$$

$$F_L^{(2)}(x,y) = -\frac{i}{2} \int_{-\ell}^{\ell} \left\{ f'(x_o) \frac{\partial F_L^{(1)}(x_{0,0})}{\partial x_o} - f(x_o) \frac{\partial^2 F_L^{(1)}(x_{0,0})}{\partial y^2} \right\} \cdot H_o^{(1)} \left(k \sqrt{(x-x_o)^2 + y^2}\right) dx_o \ . \tag{25}$$

These results are obtained by expanding the boundary condition for F_L in a power series in ε. It is easily verified that the function $\exp(ikx) + \varepsilon F_L^{(1)} + \varepsilon^2 F_L^{(2)}$ fulfills the boundary condition to order ε^2. From (24) and (25) we find

$$f_L(\theta) = \frac{k\varepsilon}{2} \int_{-\ell}^{\ell} f'(x_o)\, e^{ikx_o (1-\cos\theta)} dx_o + O\left\{(k\varepsilon)^2\right\} . \tag{26}$$

The leading term vanishes for $\theta = 0$, since $f(\pm \ell) = 0$. This is natural since $\mathrm{Re}\left\{f_L(0)\right\}$ is quadratic in ε by the forward amplitude theorem*. To obtain $f_L(0)$ we use (25) together with (24). After various integrations by parts we obtain

$$f_L(0) = \frac{-k^2 \varepsilon^2}{4\pi} \int_{-\infty}^{\infty} \frac{|\phi(t)|^2}{\sqrt{1 - t^2}} dt + o\left\{(k\varepsilon)^2\right\}, \tag{27}$$

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* See discussion following equation (17).

where

$$\phi(t) = \int_{-\ell}^{\ell} f'(x_o) \exp\left[ikx_o(1-t)\right] dx_o \; . \tag{28}$$

Here we have used the expression

$$H_o^{(1)}(|x|) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{ixt}}{\sqrt{t^2-1}} \, dt \, , \tag{29}$$

where

$$\sqrt{t^2-1} = -i\sqrt{1-t^2} \quad \text{for } |t| < 1 \; .$$

It is easy to verify that Re $\left\{f_L(0)\right\}$ is negative, while Im $\left\{f_L(0)\right\}$ is positive.

Let us now briefly discuss the solution of the reflection problem. It is well known that to obtain this solution from the cylinder solution we have to consider a grating of cylinders symmetric with respect to the plane $y = 0$, and construct the function $u(x,y;\phi) \pm u(x,y;-\phi)$, where $u(x,y;\phi)$ is the solution (for the grating of cylinders) corresponding to the incident wave $\exp\left[ik(x \cos \phi + y \sin \phi)\right]$. (The upper sign is taken when we have H||, while the lower sign refers to the case of E||.) In calculating by means of our formulas (7), (8), (9), (10) we therefore have to consider the subscript I to refer to both the directions of incidence ϕ and $-\phi$. Now if ϕ is replaced by $-\phi$ then A_L and A_R are unaffected, while, because of the symmetry, the quantity $f_I(\theta)$ for the direction of incidence $-\phi$ is equal to $f_I(-\theta)$ as calculated for the direction of incidence $+\phi$. Consequently also $f_I(0)$ and $f_I(\pi)$ remain unaltered under this transformation, whence the same is true of α_1 and α_2 of equation (10). Thus in calculating with equation (7) for the direction of incidence $-\phi$, the first term, $f_I(\theta)$, is replaced by $f_I(-\theta)$ and the remaining terms are unaltered. Consequently, for E||, the necessary subtraction leads to a simple value for the total $f(\theta)$ in the reflection problem, i.e.,

$$f_{total}(\theta) = f_I(\theta) - f_I(-\theta) \quad .$$

Since $f_I(\theta)$ is an analytic function of cos θ, it is easily verified that there is no divergence of the terms $f(\theta_\nu)/\sin\theta_\nu$ as $\theta_\nu \rightarrow 0$, in (5), so that there is no resonance phenomenon.

On the other hand, in the case of H||, we have to add the terms $f(\theta)$ and $f(-\theta)$. [It is interesting to note in this connection that, if the cylinders are taken to be of rectangular cross section and of unit or integral wavelength thickness, then it follows from our discussion of the determinants α in a previous section that $f_I(\theta) + f_I(-\theta) = 0$ if we choose $\phi = \pi/2$. In this case, therefore, our result for the reflection grating is the trivial result $\exp(iky) + \exp(-iky)$, regardless of whether the spacing is critical. However this result is easily verified for any collection of rectangular protuberances, for any spacing whatever, if $\phi = \pi/2$ and the thickness is half a wavelength. A protuberance of half a wavelength corresponds to a cylinder of thickness equal to a wavelength.]

We continue our discussion of the H|| case by specializing to the case of thin cylinders. For strips of zero thickness the far field amplitude is well known to be odd in θ. Consequently the term $f_I(\theta) + f_I(-\theta)$ will be of order kε where ε is the thickness. If we turn to the next term, however, in the case of resonance [cf. (17)], we find a different order of magnitude. For the reflection grating this term becomes in fact

$$-\frac{f_I(0)}{f_L(0)}\left[f_L(\theta) + f_L(-\theta)\right] \quad .$$

Now $f_I(0) = f_R(\phi + \pi)$ by reciprocity. For zero thickness this quantity is identically zero and for a thin strip it is of order kε. The quantity $f_L(\theta) + f_L(-\theta)$ is also of order kε. But the denominator $f_L(0)$ is of order $(k\varepsilon)^2$ [cf. (27)]. Therefore the total amplitude is a finite number, no matter now small kε is, provided we have a critical combination of incidence angle and wavelength. This is in contrast to the situation for non-critical wavelengths*. Artmann has shown also that the maximum amplitude occurs at slightly larger wavelengths than critical, and that the shift is independent of the order of the spectrum. To see this from our result we return to the formula for $f(\theta_\nu)$ at a non-critical wavelength. Consulting (7) and (9), and neglecting terms of order A_R, we find, for a case when A_L is large but not infinite,

$$(30) \qquad f(\theta_\nu) + f(-\theta_\nu) = f_I(\theta_\nu) + f_I(-\theta_\nu) + \frac{f_I(0) f_L(\theta_\nu)}{(1/A_L) - f_L(0)} \ .$$

From (12), $A_L \sim 2(kd\,\delta_a)^{-1/2}$, where, if δ is negative, the positive imaginary root is taken. Since $-\mathrm{Re}\left\{f_L(0)\right\}$ is positive by the forward amplitude theorem, there can be no cancellation between $1/A_L$ and $f_L(0)$ if δ is positive. On the other hand, if δ is negative, then $1/A_L$ is positive imaginary. In (27) we see that $\mathrm{Im}\left\{f_L(0)\right\}$ is also positive. Thus the imaginary part of the denominator will vanish at a certain negative value of δ_a, given by the formula

$$(31) \qquad \sqrt{2kd|\delta_a|} = \frac{k^2\varepsilon^2}{2\pi} \int_1^\infty \frac{|\phi(t)|^2 + |\phi(-t)|^2}{\sqrt{t^2-1}}\, dt \ .$$

Consequently [cf. (11a)] a maximum occurs for kd slightly less than critical, in general, since the numerator will only vary slightly with such a small variation in k. Equation (31) shows that δ_a is independent of the order ν, in which the maximum is being observed, and is also independent of the index p_a of the order which tends toward grazing**. Furthermore, it is easily seen that if f_c is the amplitude at the critical value, while f_m is the value at the maximum, then $(f_c - f_m)/\delta$ is of the order of f_c/δ, i.e., inversely proportional to δ. Consequently the maximum is very steep.

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* Then the leading term is $f_I(\theta) + f_I(-\theta)$, which tends to zero with the thickness.

**We keep kd fixed, and vary the angle of incidence in order to vary p_a.

It is possible to show that all these results are in agreement with the results of Artmann provided we replace Artmann's sums by integrals, in accordance with the fact that kd is to be assumed large. Using our notation, we can write Artmann's result (cf. his equation (3.32)) as

$$0 = \sqrt{\delta_a(2kd-\delta_a} + i\,\mathrm{Im}\sum_{\nu\neq p_a} \frac{d\left(\frac{2\pi k}{d}\right)^2 \left(\frac{p_a-\nu}{d}\right)^2}{k\sin\theta_\nu} \left|\varepsilon \int_{-\ell}^{\ell} f(x)\exp\left[ix\frac{\delta_a}{d} + k\cos\theta_\nu - k\right]dx\right|^2 .$$

Using the fact that d is large we can replace the above sum by an integral, via the substitution $k\cos\theta_\nu = kt$, $2\pi/kd = dt$, $-\infty < t < \infty$. Since δ_a is small we have $2\pi(\nu - p_a)/d = kt - k$, while the term δ_a/d in the exponential can be negelected. This results precisely in our formula (31). On the other hand Artmann's theory is also valid for small thickness of protuberances, even if the spacing is not large, while our results, although restricted to large spacing ($kd \gg 2\pi$), do not require that the thickness is small compared to wavelength.

We now turn to a brief comparison with new results of Twersky[12] which appear to be in disagreement with ours. To summarize this disagreement, we note that in [12] a functional equation is solved for the amplitude in the reflection grating. This functional equation is solved approximately by iteration, the justification for retaining only the leading term being the assumption of large spacing (cf. [12], equations(70) and also page 11). Then the result is a quantity $\bar{f}_{\nu o} \equiv \bar{f}(\theta_\nu, \pi - \theta_o)$ which is in turn defined as the solution of a new integral equation (cf. [12], equation (60), (56))

$$\bar{f}_{\nu o} \equiv \bar{f}(\theta_\nu, \pi-\theta_o) = f(\theta_\nu, \pi-\theta_o) - \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} f(\theta, \pi-\phi)\bar{f}(\phi, \pi-\phi_o)\,d\phi ,$$

in which the quanity f is the result based on single scattering theory, and $\bar{f}$ is to be solved for, with f as given data*. It is shown that $\bar{f}$ is, under certain circumstances, purely imaginary, and that consequently if one retains terms of higher order of magnitude in the specular amplitudes, while neglecting them in the other spectra, then energy is conserved exactly ([12], pp. 29-30). It is then emphasized in ([12], p. 30), that the results, (which according to page 29 and equation (70) are obtained for large kd) " differ significantly from a single scattering approximation in that they contain $\bar{f}$ instead of f". Arguments are given to show that the presence of f rather than $\bar{f}$ would be incorrect physically. The results insofar as they are explicit thus seem to be in disagreement with our own even before the passage to the limit of resonance, since our naive leading term is the single scattering term, in the non-critical case, and our principal result is the correction to this term, so that in the limit of infinite spacing our results reduce to single scattering. We reserve further comment until Twersky's results have appeared in final form; but it may be noted that the discrepancy would be removed if the validity of Twersky's result were additionally restricted to small scatterers, or at any rate to scatterers whose scattered amplitude is small, for then his integral equation shows that $\bar{f} \cong f$. It therefore seems possible to us that the condition for the usefulness of Twersky's iterative solution of his functional equation (70) actually dpends on the smallness of the single scattered amplitude, rather than the large kd he actually postulates. Then there would be no discrepancy in the limit of large spacing. The inherently plausible results proposed in [12], on a " heuristic" basis, for the case of one mode of propagation would also be validated in a logical sense. The reason is that for the reflection grating our suggestion would imply the assumption of the smallness of the single scattered amplitudes of a typical isolated protuberance. Since this is the case for protuberances of thickness small compared to wavelength the results should then agree with those of Artmann regardless of spacing.

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* In this formula the superscribed bar does not mean complex conjugate.

Appendix I-A

Equation (12) is proved as follows. We have from (8), (11),

$$A_{L,R} = \sqrt{\frac{2}{\pi kd}}\; e^{-i\pi/4} \sum_{1}^{\infty} \frac{\exp\left[in\,\delta_{a,b}\right]}{\sqrt{n}} \quad .$$

But the binomial formula gives

$$(1-z)^{-1/2} = \sum_{r=0}^{\infty} \frac{(-z)^r(-\frac{1}{2})!}{(-\frac{1}{2}-r)!\,r!} \quad .$$

Using the identities $\Gamma(z+1) = z\,\Gamma(z)$, $\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z) = \pi/\cos\pi z$, together with Stirling's formula

$$\Gamma(1+r) \sim \sqrt{2\pi}\; r^{r+1/2}\; e^{-r}$$

we find

$$(1-z)^{-1/2} \sim \frac{1}{\sqrt{\pi}} \sum_{1}^{\infty} \frac{z^n}{\sqrt{n}}$$

or

$$\left(1-\exp\left[i\delta_{a,b}\right]\right)^{-1/2} \sim \frac{1}{\sqrt{\pi}} \sum_{1}^{\infty} \frac{\exp\left[in\,\delta_{a,b}\right]}{\sqrt{n}}$$

and

$$A_{L,R} \sim \sqrt{\frac{2}{kd}}\; e^{-i\pi/4} \left(1-\exp\left[i\delta_{a,b}\right]\right)^{-1/2} \quad .$$

For sufficiently small $\delta_{a,b}$:

$$\left(1-\exp\left[i\delta_{a,b}\right]\right)^{-1/2} \sim (-i\sin\delta_{a,b})^{-1/2} = e^{i\pi/4}(\delta_{a,b})^{-1/2}$$

and

$$A_{L,R} \sim \sqrt{\frac{2}{kd\,\delta_{a,b}}} \quad .$$

Appendix I-B

We evaluate lim A_L/A_R by using the expressions (11a), (11b) and (13). Suppose first that ϕ is fixed and that $k \rightarrow k_c$, a given critical value. But from (11a), (11b):

$$\frac{\delta_b}{\delta_a} = \frac{kd(1+\cos\phi) - 2p_b\pi}{kd(1-\cos\phi) - 2p_a\pi} .$$

As $k \rightarrow k_c$, the numerator and denominator of the expression on the right approach zero (this is the definition of a critical frequency in case (c)). Using L'Hôpital's rule, we find

$$\lim_{k \rightarrow k_c} \frac{\delta_b}{\delta_a} = \frac{1+\cos\phi}{1-\cos\phi} = \cot^2 \frac{\phi}{2}$$

and

$$\lim_{k \rightarrow k_c} \frac{A_L}{A_R} = \lim_{k \rightarrow k_c} \sqrt{\delta_b/\delta_a} = \cot \frac{\phi}{2} .$$

In the case where k is fixed (i.e., at some value which is critical for a certain value of ϕ) while ϕ approaches the critical angle ϕ_c, we again use L'Hôpital's rule to obtain

$$\lim_{\phi \rightarrow \phi_c} \frac{\delta_b}{\delta_a} = -\frac{kd\sin\phi_c}{kd\sin\phi_c} = -1$$

and

$$\lim_{\phi \rightarrow \phi_c} \frac{A_L}{A_R} = \lim_{\phi \rightarrow \phi_c} \sqrt{\delta_b/\delta_a} = i .$$

Equation (14) is derived as follows. Substituting $\nu = p_a$ in (6), and using (11a), we have

$$k_{p_a} = k\sqrt{1-\left(1-\frac{\delta_a}{kd}\right)^2} \simeq k\sqrt{\frac{2\delta_a}{kd}} = \frac{\sqrt{2kd\,\delta_a}}{d} .$$

The case $\nu = -p_b$ is identical. Equation (15) follows from (12) and (14).

Appendix II

It is required to show that

$$\operatorname{Im} \oint_B v \frac{\partial \bar{v}}{\partial n} ds = 4 \operatorname{Re} \left\{ \bar{f}_L(0)\, a_3 \right\},$$

where

$$v = f_L(0)\, F_R(x,y) - f_R(0) F_L(x,y)$$

and B is the boundary of a given isolated cylinder.

We begin by expressing F(x,y), F(θ) (see (3), (4)) as surface integrals. The response, or scattered field, F(x,y), is given by

$$\begin{aligned} F(x,y) &= u(x,y) - u_{inc}(x,y) \\ &= \frac{i}{4} \oint_B \frac{\partial u(s)}{\partial n} H_o^{(1)}(kr_{sP})\, ds, \end{aligned}$$

where r_{sP} is the distance from an arbitrary point on B to the field point P(x,y); the integral follows from applying Green's theorem to two reduced wave equations (see [2] for details). Then

$$\begin{aligned} F_R(x,y) &= u_R(x,y) - e^{-ikx} \\ &= \frac{i}{4} \oint_B \frac{\partial u_R(s)}{\partial n} H_o^{(1)}(kr_{sP}) ds, \end{aligned}$$

$$\begin{aligned} F_L(x,y) &= u_L(x,y) - e^{ikx} \\ &= \frac{i}{4} \oint_B \frac{\partial u_L(s)}{\partial n} H_o^{(1)}(kr_{sP}) ds, \end{aligned}$$

where u_R, u_L, are the total fields resulting from the incident fields e^{-ikx}, e^{ikx} respectively. The far field form of F(x,y) is then obtained by substituting for

$H_o^{(1)}(kr_{sP})$ its asymptotic form, and suppressing higher-order terms in the integral. Applying the definition (4) we obtain

$$f_{L,R}(0) = \frac{i}{4} \oint_B \frac{\partial u_{L,R}(s)}{\partial n} \; e^{-ik\rho(s)\cos\left[\theta - \theta(s)\right]} \, ds ,$$

so that

$$f_{L,R}(0) = \frac{i}{4} \oint_B \frac{\partial u_{L,R}(s)}{\partial n} \; e^{-ikx} \, ds ,$$

$$f_{L,R}(\pi) = \frac{i}{4} \oint_B \frac{\partial u_{L,R}(s)}{\partial n} \; e^{ikx} \, ds ,$$

and

$$\bar{f}_{L,R}(0) = \frac{i}{4} \oint_B \frac{\partial \bar{u}_{L,R}(s)}{\partial n} \; e^{ikx} \, ds ,$$

$$\bar{f}_{L,R}(\pi) = \frac{i}{4} \oint_B \frac{\partial \bar{u}_{L,R}(s)}{\partial n} \; e^{-ikx} \, ds .$$

The proof of the required identity is now a direct computation. By definition

$$\begin{aligned} v &= f_L(0)\, F_R(x,y) - f_R(0)\, F_L(x,y) \\ &= f_L(0) \left(u_R - e^{-ikx}\right) - f_R(0) \left(u_L - e^{ikx}\right) . \end{aligned}$$

Since u_R, u_L vanish on B, we have

$$v(s) = -f_L(0)\, e^{-ikx} + f_R(0)\, e^{ikx}$$

and

$$\frac{\partial \bar{v}(s)}{\partial n} = \bar{f}_L(0) \frac{\partial}{\partial n} \left(\bar{u}_R - e^{ikx}\right) - \bar{f}_R(0) \frac{\partial}{\partial n} \left(\bar{u}_L - e^{-ikx}\right) ,$$

so that

$$\begin{aligned} v(s)\,\frac{\partial \bar{v}(s)}{\partial n} = & - |f_L(0)|^2\, e^{-ikx}\,\frac{\partial}{\partial n}\left(\bar{u}_R - e^{ikx}\right) \\ & - |f_R(0)|^2\, e^{ikx}\,\frac{\partial}{\partial n}\left(\bar{u}_L - e^{-ikx}\right) \\ & + f_L(0)\,\bar{f}_R(0)\, e^{-ikx}\,\frac{\partial}{\partial n}\left(\bar{u}_L - e^{-ikx}\right) \\ & + f_R(0)\,\bar{f}_L(0)\, e^{ikx}\,\frac{\partial}{\partial n}\left(\bar{u}_R - e^{ikx}\right) \quad . \end{aligned}$$

Notice that the sum of that part of the last two terms on the right which does not include $\partial\bar{u}_L/\partial n$ or $\partial\bar{u}_R/\partial n$ has an imaginary part of 0; therefore these terms may be omitted in forming

$$\operatorname{Im} \oint_B v\,\frac{\partial \bar{v}}{\partial n}\, ds \quad .$$

In the remaining terms, those which do not include $\partial\bar{u}_R/\partial n$ or $\partial\bar{u}_L/\partial n$ vanish upon integration over B. It follows that

$$\begin{aligned} \oint_B v(s)\,\frac{\partial \bar{v}(s)}{\partial n}\, ds = & - |f_L(0)|^2 \oint_B e^{-ikx}\,\frac{\partial \bar{u}_R}{\partial n}\, ds \\ & - |f_R(0)|^2 \oint_B e^{ikx}\,\frac{\partial \bar{u}_L}{\partial n}\, ds + f_L(0)\bar{f}_R(0) \oint_B e^{-ikx}\,\frac{\partial \bar{u}_L}{\partial n}\, ds \\ & + f_R(0)\,\bar{f}_L(0) \oint_B e^{ikx}\,\frac{\partial \bar{u}_R}{\partial n}\, ds \quad . \end{aligned}$$

From the definitions given above, we therefore have

$$\oint_B v \frac{\partial \bar{v}}{\partial n} ds = \frac{4}{i} \left\{ f_L(0) \, \bar{f}_R(0) \, \bar{f}_L(\pi) - |f_L(0)|^2 \, \bar{f}_R(\pi) \right\}$$

$$= \frac{4}{i} \, f_L(0) \left\{ \bar{f}_L(\pi) \, \bar{f}_R(0) - \bar{f}_L(0) \, \bar{f}_R(\pi) \right\}$$

$$= 4i \, f_L(0) \, \bar{\alpha}_3$$

where α_3 is given by (10). The result is

$$\operatorname{Im} \oint_B v \frac{\partial \bar{v}}{\partial n} ds = 4 \operatorname{Re} \left\{ \bar{f}_L(0) \, \alpha_3 \right\} .$$

A similar treatment can be given for the case of the boundary condition $\frac{\partial u}{\partial n} = 0$.

Appendix III

In case (a), the coefficients

$$\frac{2}{kd \sin \theta_\nu} \; f(\theta_\nu)$$

may be computed from (16), for $\nu \neq p_a$. If $\nu = p_a$, the denominator

$$kd \sin \theta_{p_a} = k_{p_a} d$$

approaches 0, and the value of the coefficient of index p_a is apparently indeterminate. But from the definition (6) of the spectral angles θ_ν, we have

$$\theta_\nu = \cos^{-1} \left\{ \frac{2\pi\nu}{kd} + \cos \phi \right\}$$

and from the definition (11a) of p_a, we have

$$\theta_{p_a} = \cos^{-1} (1) = 0 \; .$$

The coefficient of the term of index p_a is therefore given by

$$\frac{2}{d} \lim_{\theta_{p_a} \to 0} \frac{f(\theta_{p_a})}{k \sin \theta_{p_a}} = \frac{2}{d} \lim_{\theta_{p_a} \to 0} \frac{f(\theta_{p_a})}{k_{p_a}} = \lim_{\theta_{p_a} \to 0} A_L \, f(\theta_{p_a}) \; , \qquad .$$

where the last expression on the right follows from (15). Making use of (7) and (9), and defining for convenience

$$D = 1 - \left[A_L \, f_L(0) + A_R \, f_R(\pi) \right] + A_R A_L \, \alpha_3$$

(α_3 is defined by (10)), we find

$$A_L f(\theta_{p_a}) = \frac{1}{D} \left\{ A_L D \, f_I(\theta_{p_a}) + A_L^2 \, f_L(\theta_{p_a}) \left[f_I(0) - A_R \alpha_1 \right] + A_L A_R f_R(\theta_{p_a}) \left[f_I(\pi) - A_L \alpha_2 \right] \right\} .$$

Rearranging, we have four terms to consider, as $\theta_{p_a} \to 0$:

(i) $D^{-1} A_L f_I(\theta_{p_a})$,

(ii) $D^{-1} A_L^2 \left\{ f_I(0)\, f_L(\theta_{p_a}) - f_L(0)\, f_I(\theta_{p_a}) \right\}$,

(iii) $D^{-1} A_L A_R \left\{ f_I(\pi)\, f_R(\theta_{p_a}) - f_R(\pi)\, f_I(\theta_{p_a}) \right\}$,

(iv) $- D^{-1} A_L^2 A_R\, \beta(\theta_{p_a})$,

where

$$- \beta(\theta_\nu) = \alpha_3 f_I(\theta_\nu) - \alpha_1 f_L(\theta_\nu) - \alpha_2 f_R(\theta_\nu) .$$

Then dividing the numerators and denominators by A_L and letting $\theta_{p_a} \to 0$, we find for terms (i), (iii), the limits:

(i,a) $$\frac{f_I(0)}{A_R \alpha_3 - f_L(0)}$$

(iii,a) $$\frac{- A_R \alpha_1}{A_R \alpha_3 - f_L(0)} .$$

Proceeding similarly, we find for (ii) the limit

$$\lim_{\theta_{p_a} \to 0} \frac{\left\{ f_I(0)\, f_L(\theta_{p_a}) - f_L(0)\, f_I(\theta_{p_a}) \right\} A_L}{A_R\, \alpha_3 - f_L(0)}$$

which may be rewritten as a product of limits:

$$\frac{1}{A_R \alpha_3 - f_L(0)} \lim_{\theta_{p_a} \to 0} (A_L \sin \theta_{p_a}) \lim_{\theta_{p_a} \to 0} \frac{f_I(0)\, f_L(\theta_{p_a}) - f_L(0)\, f_I(\theta_{p_a})}{\theta_{p_a}} \lim_{\theta_{p_a} \to 0} \frac{\theta_{p_a}}{\sin \theta_{p_a}} .$$

But the first limit in the product follows from (15), the second follows from L'Hôpital's rule, and the third approaches 1. The result is that (ii) has the limit

$$\text{(ii,a)} \qquad \frac{2}{kd} \; \frac{f_I(0)\, f_L'(0) - f_L(0)\, f_I'(0)}{A_R\alpha_3 - f_L(0)} \quad .$$

Term (iv) is evaluated in the same way: using $\beta(0) = 0$, the limit is

$$\text{(iv,a)} \qquad -\frac{2}{kd} \; \frac{A_R\beta'(0)}{A_R\alpha_3 - f_L(0)} \quad .$$

Adding (i,a) - (iv,a), we obtain the stated result (21) for the limiting value of the critical term of the series (5). The corresponding analysis in case (b), where $A_R \to \infty$ and the critical term has index $\nu = -p_b$, is almost identical; and only slight modifications are required to treat case (c), where both A_L and A_R diverge and there are two critical terms.

Appendix IV

A typical cell of the grating is, say, the x-interval $(a,a+d)$ which contains the zero-th element: the boundary B of the element intersects the x-axis at $x = \alpha$, $x = \beta$, where $a < \alpha < 0 < \beta < a+d$. The total energy passing through the vertical boundaries of the cell is given by the integral of the Poynting vector P, $\int P\, ds$, taken over the lines $x = a$, $x = a+d$. Substituting for $P = E \times H$ its time-average $P_{av} = \frac{1}{2} \operatorname{Re}(E \times \overline{H})$, and calculating $\overline{H}$ in terms of E from the relevant Maxwell equation for the time-periodic field, we have as a measure of the flow of energy

$$\operatorname{Im} \int u \frac{\partial \overline{u}}{\partial n}\, ds,$$

where the integral is taken over the two lines $x = a$, $x = a+d$. But from the periodicity of the field (see (3)), we have

$$u(x+d,y) = e^{ikd \cos \phi}\, u(x,y)$$

and taking normal derivative in the outward direction, we obtain

$$\frac{\partial \overline{u}(x+d,y)}{\partial x} = - e^{-ikd \cos \phi} \frac{\partial \overline{u}(x,y)}{\partial x} .$$

It follows that the integral over the vertical boundaries of a typical cell vanishes:

$$\operatorname{Im} \int u \frac{\partial \overline{u}}{\partial n}\, ds = 0 .$$

But the field exterior to the element B is regular throughtout the cell: it follows from Green's theorem that

$$\operatorname{Im} \int_C u \frac{\partial \overline{u}}{\partial n}\, ds = \operatorname{Im} \oint_B u \frac{\partial \overline{u}}{\partial n}\, ds$$

where C consists of the vertical boundaries $x = a$, $x = a+d$, plus the horizontal boundary segments (at $y = \pm\infty$) of the cell. Since the integral over the vertical boundaries vanishes, we have

$$\text{Im} \oint_B u \frac{\partial \bar{u}}{\partial n} ds = \text{Im} \int_H u \frac{\partial \bar{u}}{\partial n} ds$$

where the integral over H (the horizontal boundary segments) measuresthe total energy flux through the cell. Now the integral over C or, equivalently, the integral over H, must vanish for an energy-conserving solution, and therefore the integral over B must vanish for an energy-conserving solution. This is trivially the case for the correct solution (which vanishes on B), but may not be for a given approximate solution. An acceptable approximation to u should conserve energy to within terms of an appropriate order (although this alone does not validate the solution), while a proposed approximate u such that $\text{Im} \oint_B u \frac{\partial \bar{u}}{\partial n} ds$ is, say, unbounded is obviously inacceptable.

For the single-scattering approximation, i.e., the assumption that

$$F(x,y) = F_I(x,y)$$

(cf., however, our approximation (7a)), we discuss the energy balance as follows. The field on B consists of the sum of four fields:

(i) the incident field u_I;

(ii) the response field F_I (which cancels the boundary value of u_I exactly);

(iii) a sum $\sum_{n=1}^{\infty} F_I(x+nd,y)$ of response fieldsfrom cylinders to the right of B;

(iv) a sum $\sum_{n=1}^{\infty} F_I(x-nd,y)$ of response fields from cylinders to the left of B.

The net field on B is therefore the sum of fields (iii) and (iv), i.e., of the approximate form (see [2], § 3, for details)

$$- A_R f_I(\pi) e^{ikx} - A_L f_I(0) e^{-ikx} .$$

Forming

$$\mathrm{Im} \oint_B w \frac{\partial \bar{w}}{\partial n} ds$$

where w is the field obtained by the single-scattering approximation, we see that energy is conserved to order $A_{R,L} = O(kd^{-1/2})$, at non-resonance frequencies but is not conserved at resonance frequencies, since

$$w \cong - A_R f_I(\pi) e^{ikx} - A_L f_I(0) e^{-ikx}$$

on B and either A_R or A_L or both diverge at resonance.

In the present approximation, the field on B is again the sum of four fields, as follows:

(i) the incident field u_I;

(ii) the response field F(x,y) (cf. (7a));

(iii) a sum $\sum_1^{\infty} H_o^{(1)}(kr_n) g(r_n,\theta_n)$, i.e., of cylindrical waves of amplitudes $g(r_n,\theta_n) \equiv \left[f(\theta_n) + \frac{f_1(\theta_n)}{r_n} + \frac{f_2(\theta_n)}{r_n^2} + \dots \right]$. Here (r_n,θ_n) are coordinates of the field point, measured from the origin (x+nd,0) of response fields from cylinders to the right of B (in terms of our previous notation we are representing F(x-nd,y) as an asymptotic series in decreasing powers of r_n);

(iv) a sum $\sum_{-\infty}^{-1} H_o^{(1)}(kr_n) g(r_n,\theta_n)$ of response fields from cylinders to the left of B.

The analysis leading to (7a) involved replacing $H_o^{(1)}(kr_n)$ by the first term in its asymptotic expansion, and replacing $g(r_n,\theta_n)$ by an amplitude $f(\theta_n)$ independent of r_n:

(A.1) $$H_o^{(1)}(kr_n) \sim \sqrt{\frac{2}{\pi kr_n}} \exp\left[i(kr_n - \pi/4)\right] ,$$

(A.2) $$g(r_n,\theta_n) \sim f(\theta_n) .$$

This amounts to replacing the sum of fields (iii), (iv) by

$$A_R\, f(\pi)\, e^{ikx} + A_L\, f(0)\, e^{-ikx} .$$

Adding (i), we see that the sum of fields (i), (iii), (iv), is exactly cancelled by (ii); i.e., the net field on B vanishes, to within the degree of approximation represented by (A.1) and (A.2).

Introducing higher-order terms in the expansions of fields (iii), (iv) will of course lead to a non-vanishing field on B: it remains to be shown that energy [as measured by $\operatorname{Im} \oint_B u\,(\partial\bar{u}/\partial n')ds$] is conserved to within $o\left((kd)^{-1/2}\right)$. We accordingly introduce the full asymptotic expansion of $H_o^{(1)}(kr_n)$ in place of (A.1), and an expansion in powers of $(r_n)^{-1}$ in place of (A.2), viz:

(B.1) $$H_o^{(1)}(kr_n) \sim \sqrt{\frac{2}{\pi kr_n}} \exp\left[i(kr_n - \pi/4)\right] \left\{1 + \sum_{k=1}^{\infty} \frac{(-1)^k a_k}{(2ir_n)^k}\right\},$$

where

$$a_k = \frac{(-1)^{2k-1}(1^2)(3^2)\ldots(2k-1)^2}{2^{2k}\ k!} ,$$

(B.2) $$g(r_n,\theta_n) \sim f(\theta) + \frac{f_1(\theta)}{r_n} + \frac{f_2(\theta)}{(r_n)^2} + \ldots .$$

The functions $f_1(\theta)$, $f_2(\theta)$, ... can be deduced from $f(\theta)$; and if we define functions $f_{I,1}(\theta)$, $f_{L,1}(\theta)$, $f_{R,1}(\theta)$, ... it follows that each of these has the same form (cf. (7)) as $f(\theta)$. Using (B.2) instead of (A.2), attending for the moment to the second term of (B.2) and approximating r_n by nkd, we obtain for (iv) an expansion whose leading term is a sum of terms of type $(nkd)^{-3/2}$, e.g.,

$$f_1(0) \sum_1^{\infty} \frac{e^{inkd(1-\cos \phi)}}{(nkd)^{3/2}} \quad .$$

This series converges absolutely, even at critical frequencies, and an argument of the type given in Appendix III shows that $f_1(0)$ is finite at critical frequencies. Therefore the use of (B.2) in place of (A.2) yields conservation of energy to within $o\left[(kd)^{-1/2}\right]$.

It follows that we need only consider further terms of $\sum_1^{\infty} H_o^{(1)}(kr_n)f(\theta_n)$. Applying (B.1), and using

$$r_n \simeq (x + nd) + \frac{y^2}{2nd}$$

$$\theta_n \simeq \frac{y}{nd}$$

$$f(\theta_n) \simeq f(0) + \frac{y}{nd} f'(0)$$

we again obtain a series which is dominated by $\sum_1^{\infty} (nkd)^{-3/2}$, multiplied by terms which remain finite at critical frequencies.

Appendix V

The perturbation calculation leading to (24), (25), is as follows. According to (23), the boundary condition $\partial H/\partial n = 0$ on the magnetic field has the form

$$0 = \frac{\partial}{\partial n}\left\{e^{ikx} + F_L(x,y)\right\}$$

$$= \frac{\partial e^{ikx}}{\partial n} + \frac{\partial}{\partial n}\left\{\varepsilon\, F_L^{(1)}(x,y) + \varepsilon^2\, F_L^{(2)}(x,y) + \dots\right\} ,$$

where the normal derivatives are evaluated on $y = \pm\, \varepsilon\, f(x)$; or since the symmetry of the problem permits us to work in the half-space $y \geq 0$, we will take the normal derivatives on $y = +\, \varepsilon\, f(x)$. Then

$$\frac{\partial H}{\partial n} = \nabla H \cdot \frac{d\bar{n}}{|dn|} = H_y \frac{dx}{ds} - H_x \frac{dy}{ds}$$

$$= \frac{H_y}{\sqrt{1 + \varepsilon^2 \left[f'(x)\right]^2}} - \frac{H_x\, \varepsilon\, f'(x)}{\sqrt{1 + \varepsilon^2 \left[f'(x)\right]^2}} .$$

The boundary curve $y = \varepsilon\, f(x)$ is assumed to satisfy two conditions:

(a) $f'(0) = 0$ at the leading and trailing edges, i.e., the curve fits smoothly into the x-axis (under this condition, there are no singularities in the analysis);

(b) the expansion

$$\left[1 + \varepsilon^2\left[f'(x)\right]^2\right]^{-1/2} = 1 - \frac{\varepsilon^2}{2}\left[f'(x)\right]^2 + \dots ,$$

is valid, i.e., $f'(x)$ is finite.

Next, evaluating H_x, H_y, on the boundary, we find (to within terms of order ε^3)

$$\frac{\partial H}{\partial y} = \frac{\partial H(x, \varepsilon\, f(x))}{\partial y}$$

$$= \varepsilon \, \frac{\partial F_L^{(1)}(x,0)}{\partial y} + \varepsilon^2 \left\{ \frac{\partial F_L^{(2)}(x,0)}{\partial y} + f(x) \, \frac{\partial^2 F_L^{(1)}(x,0)}{\partial y^2} \right\} ,$$

$$\frac{\partial H(x, \varepsilon f(x))}{\partial x} = ik\, e^{ikx} + \varepsilon \, \frac{\partial F_L^{(1)}(x,0)}{\partial x} + \varepsilon^2 \, \frac{\partial F_L^{(2)}(x,0)}{\partial x} .$$

Using condition (b) above, we find

$$\frac{H_y}{\sqrt{1 + \varepsilon^2 |f'(x)|^2}} = \varepsilon \, \frac{\partial F_L^{(1)}(x,0)}{\partial y} + \varepsilon^2 \left\{ \frac{\partial F_L^{(2)}(x,0)}{\partial y} + f(x) \, \frac{\partial^2 F_L(x,0)}{\partial y^2} \right\},$$

$$\frac{H_x \varepsilon\, f'(x)}{\sqrt{1 + \varepsilon^2 |f'(x)|^2}} = \varepsilon\, f'(x) \; ik\, e^{ikx} + \varepsilon^2 \, f'(x) \, \frac{\partial F_L^{(1)}(x,0)}{\partial x} .$$

The boundary condition $\partial H/\partial n = 0$ therefore yields the two conditions (obtained by setting the coefficients of ε, ε^2, respectively, equal to 0):

$$\frac{\partial F_L^{(1)}(x,0)}{\partial y} - f'(x) \left(ik\, e^{ikx} \right) = 0,$$

$$\frac{\partial F_L^{(2)}(x,0)}{\partial y} + f(x) \, \frac{\partial^2 F_L^{(1)}(x,0)}{\partial y^2} - f'(x) \, \frac{\partial F_L^{(1)}}{\partial x} = 0 .$$

Since the quantitites $F_L^{(1)}(x,y)$, $F_L^{(2)}(x,y)$ must satisfy the reduced wave equation, they are determined as the solutions of boundary-value problems by these two conditions.

The boundary-value problem for $F_L^{(1)}(x,y)$ is:

$$\frac{\partial^2 F_L^{(1)}}{\partial x^2} + \frac{\partial^2 F_L^{(1)}}{\partial y^2} + k^2 F_L^{(1)} = 0,$$

$$\frac{\partial F_L^{(1)}}{\partial y} = ik\, e^{ikx} f'(x), \qquad |x| = \ell$$

$$= 0 , \qquad |x| > \ell \quad .$$

Applying Green's theorem over $y = \varepsilon f(x)$, we have

$$\int \left\{ F_L^{(1)} \Delta\, G - G \Delta\, F_L^{(1)} \right\} dS = \oint \left\{ F_L^{(1)} \frac{\partial G}{\partial n} - G \frac{\partial F_L^{(1)}}{\partial n} \right\} ds$$

(where G is the appropriate Green's function), or

$$F_L^{(1)}(x,y) = - ik \int_{-\ell}^{\ell} e^{ikx_o} f'(x_o)\, G\, ds \quad .$$

The appropriate Green's function (for $y \geq 0$), whose normal derivative vanishes on the strip, is

$$G = \frac{i}{2} H_o^{(1)} \left(k \sqrt{(x-x_o)^2 + (y-y_o)^2} \right) \quad .$$

Substituting the value of G at $y_o = 0$, we verify (24):

$$F_L^{(1)}(x,y) = \frac{k}{2} \int_{-\ell}^{\ell} e^{ikx_o} f'(x_o)\, H_o^{(1)} \left(k \sqrt{(x-x_o)^2 + y^2} \right) dx_o \quad .$$

Solving the boundary-value problem

$$\frac{\partial^2 F_L^{(2)}}{\partial x^2} + \frac{\partial^2 F_L^{(2)}}{\partial y^2} + k^2 F_L^{(2)} = 0,$$

$$\frac{\partial F_L^{(2)}(x,0)}{\partial y} = f'(x) \frac{\partial F_L^{(1)}}{\partial x} - f(x) \frac{\partial^2 F_L^{(1)}(x,0)}{\partial y^2} \quad , \qquad |x| < \ell$$

$$= 0 \quad , \qquad |x| > \ell$$

in the same way, we obtain (25):

$$F_L^{(2)}(x,y) = -2\int_{-\ell}^{\ell} G\,\frac{\partial F_L^{(2)}(x,0)}{\partial y}\,dx_o$$

$$= -\frac{i}{2}\int_{-\ell}^{\ell}\left\{f'(x_o)\,\frac{\partial F_L^{(1)}(x,0)}{\partial x} - f(x_o)\,\frac{\partial^2 F_L^{(1)}(x_o,0)}{\partial y^2}\right\} H_o^{(1)}\left(k\sqrt{(x-x_o)^2+y^2}\right)dx_o.$$

To obtain (26), we recall (cf. (4)) that $f_L(\theta)$ is the coefficient of

$$\sqrt{\frac{2}{\pi R_n}}\;e^{i(kr-\pi/4)}$$

in the far field form of $F_L(x,y) = \varepsilon F_L^{(1)}(x,y) + \varepsilon^2 F_L^{(2)}(x,y)$. Using

$$H_o^{(1)}(k\rho) \sim \sqrt{\frac{2}{\pi k\rho}}\;e^{i(k\rho-\pi/4)},$$

where

$$\rho = \sqrt{(x-x_o)^2+y^2} = r\sqrt{1-\frac{2xx_o}{r^2}+\frac{x_o^2}{r^2}}$$

$$\simeq r\left\{1-\frac{xx_o}{r^2}+\frac{x_o^2}{2r^o}\right\}$$

$$\simeq r - \frac{(r\cos\theta)x_o}{r}$$

$$= r - x_o\cos\theta\;,$$

and approximating $\rho^{-1/2}$ by $r^{-1/2}$, we have from (24), (25):

$$f_L(\theta) = \frac{k\varepsilon}{2}\int_{-\ell}^{\ell}\exp\left[ikx_o(1-\cos\theta)\right]f'(x_o)dx_o + O\left((k\varepsilon)^2\right).$$

References

[1] Karp, S. - Diffraction by finite and infinite gratings; Bulletin Amer. Phys. Soc., Phys. Rev., 86, p. 586, 1952.

[2] Karp, S. - Diffraction by an infinite grating of cylinders; New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, Research Report No. EM-85, October, 1955.

[3] Karp, S. - Scattering by a combination of obstacles; McGill University Symposium, June, 1953.

[4] Karp, S. and Russek, A. - Diffraction by a wide slit; New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, Research Report No. EM-75, Feb., 1955.

[5] Twersky, V. - Multiple scattering of radiation, Part II (the grating); New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, Research Report No. EM-39, December, 1951.

[6] Twersky, V. - Multiple scattering of radiation, Part II (the grating); J.A.P., 23, 1099 (1952).

[7] Wood, R.W. - Anomalous diffraction gratings; Phys. Rev., 48, 928 (1935).

- Diffraction gratings with controlled groove form and abnormal distribution of intensity; Phil. Mag., 23, 310 (1912).

- On a remarkable case of uneven distribution of light in a diffraction grating spectrum; Phil. Mag., 4, 396 (1902).

[8] Lord Rayleigh - On the dynamical theory of gratings; Proc. Roy. Soc., London (A) 79, 399 (1907).

[9] Fano, U. - Zur Theorie der Intensitätsanomalien der Beugung; Ann. der Physik (5), 32, 393 (1938).

[10] Strong, J. - Effect of evaporation films on energy distribution in grating spectra; Phys. Rev., 39, 291 (1936).

[11] Artmann, K. - Zur Theorie des anomalen Reflexion von Optischen Strichgittern; Z. für Physik, 119, 529 (1942).

[12] Twersky, V. - Scattering of waves by an infinite grating; Electronics Defense Laboratory, Mountain View, California, Eng. Rep. No. EDL-E5, April (1955).

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